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Current discussion of papers sponsored by the Engineering Mechanics Division is presented as follows:

Number		Page
177	"Earthquake Stresses in Shear Buildings" (Published in March, 1953; discussion closed)	
	Blume, John A.	1
	Masur, E. F.	8
	Cohen, Edward; Laing, Edward; Levy, Leon S.	11
	Salvadori, Mario G. (Closure)	12
179	"Stiffness Charts for Gusseted Members Under Axial Load" (Published in March, 1953; discussion closed)	
	Stevens, Leonard K.	15
	Goldberg, John E. (Closure)	16
202	"The Application of Heaviside's Step-Function to Beam Problems" (Published in July, 1953; discussion closed)	
	van Langendonck, T.	17
223	"Applications of the Relaxation Technique in Fluid Mechanics" (Published in July, 1953; discussion closed)	
	Corrections	20
258	"Mechanics of Manifold Flow" (Published in August, 1953; discussion closes December 1, 1953)	
	Corrections	20

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JOHN A. BLUME,[16] M. ASCE.—There has been considerable technical literature concerning the analysis of buildings subjected to earthquakes. A great deal of this revived interest in building dynamics has undoubtedly been induced by the possible application of earthquake-resistant design methods to blast design. Such renewed interest is heartily welcomed. A word of caution is required, however, regarding the general acceptance of mathematical or numerical methods based on idealized structures as constituting complete earthquake-resistant design. Caution is especially necessary when the buildings are conveniently subjected, for the purpose of analysis, to sinusoidal motion of specific characteristics, or even to a few cycles of sinusoidal motion of different characteristics.

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The author's contribution can be considered another method of investigating building dynamics, but neither this paper, that by C. S. Whitney, M. ASCE, B. G. Anderson, and the author,[15] nor any other idealized mathematical treatment should be thought to constitute a panacea. In this regard, reference is made to the statement in Mr. Salvadori's conclusions:

[15] "A Comprehensive Numerical Method for the Analysis of Earthquake Resisting Structures," by C. S. Whitney, B. G. Anderson, and M. G. Salvadori, *Journal*, A.C.I., Vol. 23, No. 1, September, 1951.

> "The availability of such procedures should encourage the designer to abandon the simplified and unrealistic methods of analysis suggested by codes (such as the 0.1g method) with obvious gains in economy and safety."

The mere assumption of static lateral forces, whether based on 0.1g or any other percentage of g, is an unrealistic approach to a dynamic problem. Building codes, including the San Francisco (Calif.) code and the 1952 Uniform Building Code,[17] vary the seismic factor with the height or number of stories.

[17] "Uniform Building Code," Pacific Coast Building Officials Conference, Los Angeles, Calif., Vol. I, 1952, p. 280.

A suggested code[18] approaches the problem from a more realistic viewpoint—permitting decreases in lateral design forces with flexibility and considering actual damage and earthquake records. It is untrue that the use of assumed static lateral forces in design is entirely or even largely caused by the lack of

[18] "Lateral Forces of Earthquake and Wind," *Transactions*, ASCE, Vol. 117, 1952, Appendix I.—Lateral Force Code, p. 746.

numerical procedures for dynamic analysis. There are actually many methods and techniques—mathematical, experimental, and empirical. Two mathematical methods which can be used to obtain building periods in much less time than the method suggested by Mr. Salvadori will be presented.

Some of the factors involved in the design of buildings to resist earthquakes greatly complicate and make questionable the implied accuracy and security of any rigorous analysis as a sole criterion. These factors are:

1. The motion of the earth in the general area of the greatest energy release is not a simple sinusoidal movement or even a superimposition of only a few such waves of variable period, amplitude, and acceleration. The motion is violent, chaotic, and in three directions. It may last from 30 sec to 40 sec or more. The motion is exceedingly complex and variable.
2. The epicenter is not known in advance, although active faults usually are.
3. There are many active faults that may affect structures in some localities. The assumption, therefore, that distant shocks may transmit smoother waves of lesser intensity (but more subject to analysis) can be dangerous as the only basis for design.
4. The natural periods of the building, which are greatly affected by the walls, fireproofing, ground conditions, and adjacent structures, are subject to change during an earthquake. With increased amplitude the periods shorten as more partitions, walls, and other items come into structural action, and the periods lengthen as cracks and failure occur.
5. The damping is not only conveniently viscous but is frictional and varies not only with materials and types of construction but with amplitude and damage.
6. The concept of using dominant ground periods or important Fourier components of actual records as disturbing forces or motions is subject to more exploration before being a reliable basis for design. Moreover, such characteristics may be not only local, but local for a specific origin of energy release, which point may not be the same in future shocks.

There are several other factors complicating rigorous aseismic design. This approach should be joined with (a) the study of actual damage (and lack of damage) in exposed structures; (b) field research; (c) dynamic laboratory research (shaking table and similar equipment); (d) mathematical research; (e) obtaining and analyzing reliable earthquake records, particularly for the response of various systems of one and more masses to such motion, with various damping coefficients, and spectrum studies; (f) practical design considerations; (g) practical considerations of function, economy, and risk; (h) complete drawings showing details; (i) adequate construction supervision; (j) code requirements; and (k) committee work and discussion.

The writer has been engaged in all of these phases of the problem and wishes to defend (at the present state of knowledge) modern (1953) methods using assumed lateral forces for codes. It is suggested that such codes be constantly improved as experience and knowledge indicate. The static lateral forces can be, and to some extent have been, based on dynamics and mathematics as well as on many other considerations. Special structures are often designed dynamically. Earthquake-resistant design is essentially an art which uses either

assumed static forces or dynamic forces (or both) as a means to an end. No code, mathematical method, or assumed force can remove the basic requirement for sound, practical judgment and experience. Above all, continued research is needed.

Research was performed at Stanford University at Stanford, Calif., in 1933 with Lydik S. Jacobsen as faculty sponsor.[19] This work disclosed many interesting factors concerning the dynamic characteristics of a building and the methods of computing these characteristics. The results of this research were not widely distributed because they were considered interesting but academic in view of the many other factors in the design problem. Many of the writer's comments on the author's paper are based on that research.

[19] "The Reconciliation of the Computed and Observed Periods of Vibration of a Fifteen-Story Building," by John August Blume and Harry Leonard Hesselmeyer, thesis presented to Stanford University, at Stanford, Calif., in 1934, in partial fulfilment of the requirements for the degree of Engineer.

Shear versus Flexure.—The natural vibration frequencies of a hypothetical five-story structure have been developed by Mr. Salvadori from generalized equations based on certain assumptions. There is little error involved in the assumption that the mass of each story is concentrated at the corresponding floor level when four or more stories are involved. This assumption was the construction basis of dynamic building models at Stanford University by Mr. Jacobsen. However, in the introduction the author states that a structure with an height-to-width ratio of 5 or less can be termed a "shear building," and the influence of direct stresses (flexure of the building as a whole) can be neglected. The listed reference[2] does not reconcile the difference between wind and earthquake design. Normal wind design for framed buildings considers frame resistance to all wind forces, whereas earthquake design usually considers not only the frame, but fireproofing, walls, and other structural elements—in proportion to their relative rigidities. Of course, there are some exceptions in both cases. If the walls and fireproofing are neglected, the shear distortion between level floors is much greater than the deflection caused by direct stress in the columns, and the 5-to-1 ratio applies. In earthquake design, however, the walls, floors, fireproofing, permanent partitions, and shear walls provide not only mass but rigidity and strength. Many practicing engineers are convinced of the value of a good frame as a secondary defense,[18] but they do not neglect the walls or other structural elements which can provide both strength and rigidity and which may constitute a hazard to personnel.

[2] "Wind Bracing," by H. V. Spurr, McGraw-Hill Book Co., Inc., New York, N. Y. 1930.

An actual fifteen-story building was analyzed in detail at Stanford University.[19] The height-to-width ratio in one direction was 3.27. The fundamental period computed for this same direction with shear alone (neglecting flexure and ground movement) was 0.605 sec as compared to 1.026 sec if shear and flexure were both considered. The error of the former is 70%. When ground flexibility is introduced the period becomes 1.32 sec. In the other direction, the height-to-width ratio was 2.85 and the fundamental period with shear alone was 0.764 sec as compared to 1.04 sec with shear and flexure; the error was therefore 36%. With ground movement in this direction, T_1 was 1.23 sec. An error in T_2 for the same building caused by neglecting flexure would be from 15% to 20%. The walls of this building contained a considerable number of windows—about 47% of the wall length in the first direction and 60% in the other direction.

For buildings with an height-to-width ratio of 5, and with other slenderness ratios, flexure as a whole should be considered in the determination of the first, second, and possibly the third modes.

The First Mode.—The first mode of an hypothetical building is evaluated in Table 4 by the Stodola-Vianello method. Other methods give results with less expenditure of work and time and with greater ease of introducing factors such as flexure, ground rotation, and ground translation. One simple solution for the first mode for any number of different masses and for different spring constants can be made directly without trial and with only elementary mathematics; the Lord Rayleigh potential energy method is applied to d'Alembert's principle and the basic equations of motion. Although mathematically approximate, the solution can be considered sufficiently accurate for engineering purposes, which involve many factors for which the value of the second significant figure is doubtful.

This method will be demonstrated by using the same geometric and elastic properties as set forth in Table 1. It is assumed that the dynamic deflection curve for this mode is the same as the static deflection curve of the system loaded laterally by its own weight. Table 9 illustrates the entire process and

TABLE 9.—DIRECT COMPUTATION OF APPROXIMATE FIRST-MODE FREQUENCY

i	M_i	$W = M_i g$	ΣW	$K_i \times 10^{-3}$	$d\Delta_s = \frac{\Sigma W}{K_i}$	Δ_s	Δ_s^2	$W \Delta_s$	$W \Delta_s^2$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
5	11.65	4,502	4,502	6.84	0.658	8.478	71.88	38,168	323,604
4	15.53	6,001	10,503	7.98	1.316	7.820	61.15	46,928	366,961
3	15.53	6,001	16,504	9.12	1.810	6.504	42.30	39,031	253,842
2	15.53	6,001	22,505	10.26	2.193	4.694	22.03	28,169	132,202
1	15.53	6,001	28,506	11.40	2.501	2.501	6.26	15,009	37,506
								$\Sigma W \Delta_s =$ 167,305	$\Sigma W \Delta_s^2 =$ 1,114,115

has an extra column, M_i, which would ordinarily be eliminated since W would be entered directly in pounds or kips. The equation,

$$\omega_1^2 = g \frac{\Sigma W_i \Delta_i}{\Sigma W_i \Delta_i^2} \quad \ldots\ldots (71)$$

can easily be proved valid by equating the maximum potential energy in the fully deflected position to the maximum kinetic energy when vibrating through the static equilibrium position. It is simple to add the flexural deflection, Δ_f, and ground rotation values, Δ_g, to the shear deflections before computing $W \Delta$ and $W \Delta^2$.

From Table 9 and Eq. 71

$$\omega_1^2 = 386.4 \times \frac{167{,}305}{1{,}114{,}115} = 58.025 \quad \ldots\ldots (72a)$$

from which

$$\omega_1 = 7.617 \quad \ldots\ldots (72b)$$

The value of ω_1 computed in one step is 0.79% in error from the value (7.557) determined from Eq. 5. This error from the classical solution is much less than the knowledge of the actual modulus of elasticity of concrete or steel.

It should be noted that ω_1 ("frequency" in the author's nomenclature) is actually the angular velocity in radians per second. The frequency is $\frac{\omega_1}{2\pi}$. The first period, T_1, is $\frac{2\pi}{\omega_1}$, which is 0.824 sec according to results of Table 9.

Fig. 3 is a qualitative plot of the first-mode curve, wherein the values obtained from Table 9 are reduced in proportion to the maxima obtained by the two methods.

A more accurate method of computing ω_1 and all natural frequencies involves a trial type of solution, but less labor than Mr. Salvadori's procedure. This procedure[19] will be demonstrated subsequently. It was adapted to building dynamics (apparently for the first time) from a device first given by H. Holzer[20] and applied to the torsional vibration problem of a shaft by S. Timoshenko.[21].

[19] "Vibration Problems in Engineering," by S. Timoshenko, D. Van Nostrand Co., Inc., New York, N. Y., 2d Ed. 1937, p. 263.

[20] "Die Berechnung der Drehschwingungen," by H. Holzer, J. Springer Verlag, Berlin, 1921.

General Method for Shear Vibration.—For convenience and flexibility in introducing variable ground characteristics in the research at Stanford University,[19] the top story (in lieu of the lowest story) was called 1 in the i series and the nth story the lowest. In order to avoid confusion with the author's notation, the notation z is used instead of i (see Fig. 4).

From d'Alembert's principle,

$$M_z \ddot{x}_z + K_z (x_z - x_{z+1}) - K_{z-1} (x_{z-1} - x_z) = 0 \quad \ldots\ldots (73)$$

and by the addition of the n equations of motion.

$$\sum_1^n M_z \ddot{x}_z + K_n (x_n) = 0 \quad \ldots\ldots (74)$$

Assuming simple harmonic motion (for the natural system, not the forcing distrubance),

$$x_z = X_z \sin \omega t \quad \ldots\ldots (75)$$

and

$$\ddot{x}_z = - X_z \omega^2 \sin \omega t \quad \ldots\ldots (76)$$

Substituting and letting $\sin \omega t$ equal unity, for maximum motion,

$$- \sum_1^n M_z \omega^2 X_z + K_n X_n = 0 \quad \ldots\ldots (77)$$

Also, for the zth mass it can be shown that

$$X_z = X_{z-1} - \frac{1}{K_{z-1}} \sum_1^{z-1} M \omega^2 X \quad \ldots\ldots (78)$$

The process can best be undertaken in tabular form as in Table 10, which uses the same geometric and elastic values established by the author.

TABLE 10.—EXAMPLE OF FREQUENCY CALCULATION WITH ω ASSUMED EQUAL TO 20.61

i	M	$M\omega^2$	X	$M\omega^2 X$	$\Sigma M\omega^2 X$	$\frac{1}{K} \times 10^3$	$\frac{1}{K}\Sigma M\omega^2 X$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	11.65	4948.6	1.000	4948.6	4948.6	0.1462	0.7235
4	15.53	6596.7	0.2765	1824.0	6772.6	0.1253	0.8486
3	15.53	6596.7	−0.5721	−3774.0	2998.6	0.1006	0.3286
2	15.53	6596.7	−0.9007	−5941.6	−2943.0	0.0975	−0.2869
1	15.53	6596.7	−0.6138	−4049.1	−6992.1	0.0877	

The first step in this process is to assume an ω-value and then to assume that X is equal to unity. Since frequencies of such a system are independent of amplitude, the X-values can later be proportionally adjusted, if desired. At a solution for an harmonic period, Eq. 77 must be satisfied.

$$6992.1 + 11{,}400\ (-0.6138) = -5 \approx 0 \qquad (79)$$

In this case (in Table 10) the solution is for the second mode since the criterion is essentially met and the X-values indicate two nodes, or second-mode movement. The procedure is to select various values of ω and to plot the results. With a few trials, which take little time with a calculating machine, solutions for frequency as well as mode shapes are readily obtained.

The curve in Fig. 5 is the curve of Eq. 77 for the author's building. It is ordinarily unnecessary, of course, to plot the curve except near the desired frequency.

It should be noted in Fig. 5 that each crossing of the vertical zero axis represents a natural-mode frequency. All those shown agree with those computed by Mr. Salvadori after a more laborious process. In Fig. 3 the shape of the fundamental mode coincides exactly with the curve obtained from Eq. 6b. The X-values of Table 10, when proportioned to the author's top-story value, agree with his second-mode values of Table 3(a).

The mathematical research undertaken at Stanford University did not limit the computation of the first or second modes to shear only, but included flexure of the building as a whole, ground rotation, ground translation, the effectiveness of brick filler walls in conjunction with the steel frame, and other considerations. Actually the computation methods were means to the end of learning how multi-story buildings vibrate. It has been most interesting to substitute Mr. Salvadori's hypothetical building values in the equations and in the methods developed in 1933, and to obtain the same results as did the author.

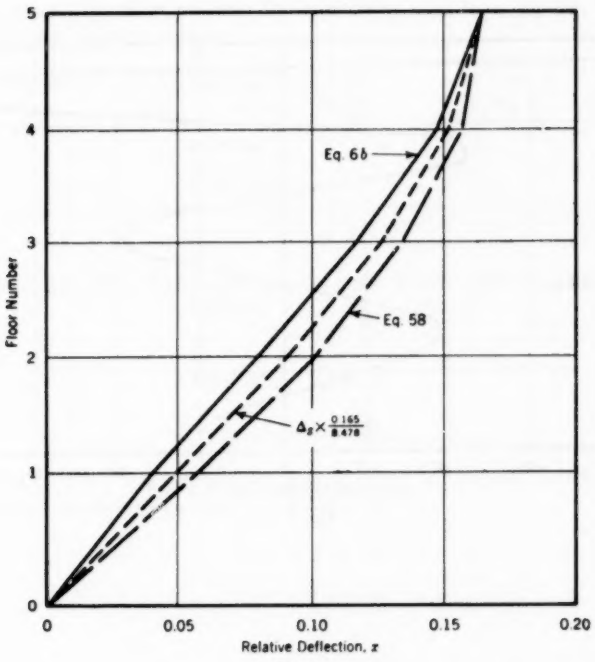
Floor Number
Relative Deflection, x
Eq. 6b
Eq. 58
$\Delta_x \times \frac{0.165}{8.478}$
0
1
2
3
4
5
0
0.05
0.10
0.15
0.20

FIG. 3

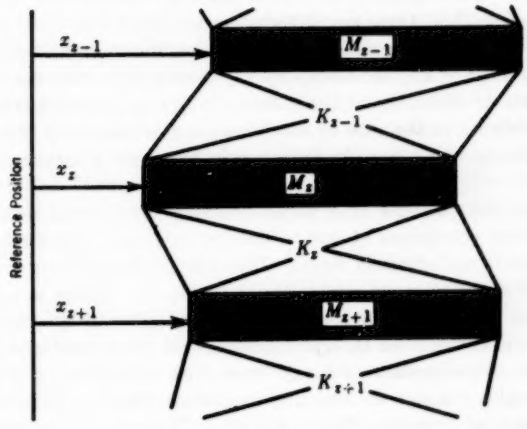
Reference Position
x_{z-1}
x_z
x_{z+1}
M_{z-1}
M_z
M_{z+1}
K_{z-1}
K_z
K_{z+1}

FIG. 4

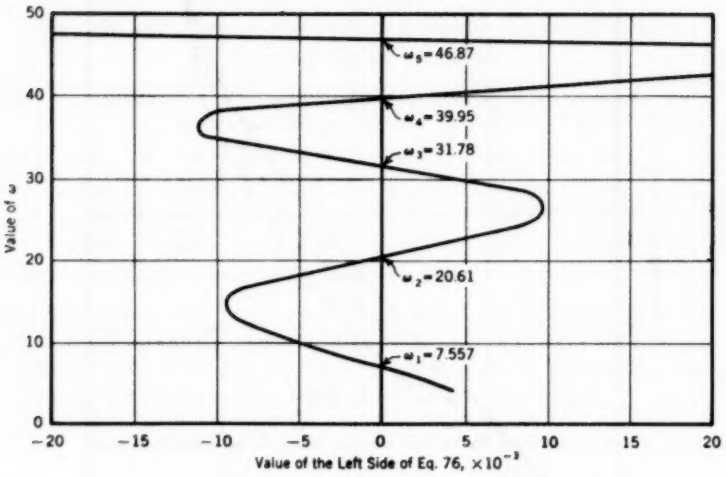

FIG. 5

E. F. MASUR.[22]—It is gratifying to note that Mr. Salvadori's paper marks a return to the "classical" procedure of determining the dynamic response of an elastic system to a set of applied forces by expanding that response in terms of the natural modes of vibration of the system. This approach has the further advantage (in addition to the ease of computation mentioned by the author) of permitting the study of various loading conditions with a minimum of additional effort.

[22] Associate Prof. of Civ. Eng., Illinois Inst. of Technology, Chicago, Ill.

It is felt that the concept of a shear building needs further elucidation. Other than the two conditions stated in the Introduction, the author also assumes that the floors are infinitely rigid. This simplification leads to the almost diagonal form of the frequency determinant in Eq. 5. There is some doubt, however, whether the error thus introduced is negligible in the case of actual structural-design practice. In fact, the existence of joint rotations lessens the resistance of a structure against sidesway to such an extent that it is reasonable to expect appreciably smaller natural frequencies of vibration than those based on the idealization of complete floor rigidity. This weakening effect is well recognized in conventional wind-bracing analysis, in which the stiffness of the floor girders is as much a factor as that of the columns.

To illustrate this point numerically the simple one-bay, three-story frame shown in Fig. 6 should be considered. The mass, modulus of elasticity, and moment of inertia are, respectively, M, E, and I for the columns, and 8 M, E, and 4 I for the girders. When the distributed spandrel masses are combined

with the respective floor masses, the frequency equation, on the assumption of rigid girders, takes the form:

$$\begin{vmatrix} (48\,D - 10\,\omega^2) & -24\,D & 0 \\ -24\,D & 48\,D - 10\,\omega^2 & -24\,D \\ 0 & -24\,D & 24\,D - 9\,\omega^2 \end{vmatrix} = 0 \quad \ldots\ldots (79a)$$

in which

$$D = \frac{E\,I}{M\,L^3} \quad \ldots\ldots (79b)$$

The roots of Eq. 79*a*—that is, the natural frequencies of the idealized structure, are

$$\omega^2_1 = 0.50\,D \quad \ldots\ldots (80a)$$

$$\omega^2_2 = 3.84\,D \quad \ldots\ldots (80b)$$

and

$$\omega^2_3 = 7.92\,D \quad \ldots\ldots (80c)$$

These results differ from the exact values (that is, "exact" within the other two assumptions regarding shear buildings) of

$$\omega^2_1 = 0.31\,D \quad \ldots\ldots (81a)$$

$$\omega^2_2 = 2.81\,D \quad \ldots\ldots (81b)$$

and

$$\omega^2_3 = 7.17\,D \quad \ldots\ldots (81c)$$

It can be seen that the nonrigidity of the floors affects the magnitude of the natural frequencies of vibration to a significant extent. Furthermore, it can be shown that the effect on the modes is as great, or greater, than the effect on the natural frequencies of vibration.

It is possible, however, to make some allowance for the joint rotations without sacrificing the simplicity of the form of the author's equations of motion. This can be done by assuming that all columns and girders display a point of contraflexure at their midpoints. In that case, the conventional analysis shows that the spring constant K_i, as used by Mr. Salvadori, must be multiplied by a factor,

$$c = \frac{\sum_G S}{\sum_M S} \quad \ldots\ldots (82)$$

in which $\sum_G$ extends over all the girders and $\sum_M$ extends over all the members (that is, girders and columns) meeting at a joint. In Eq. 82, S is used to denote the stiffness of a member.

If this procedure is applied to the example under consideration, the frequency equation is modified to

$$\begin{vmatrix} (30\,D - 10\,\omega^2) & -12\,D & 0 \\ -12\,D & (26\,D - 10\,\omega^2) & -14\,D \\ 0 & -14\,D & 14\,D - 9\,\omega^2 \end{vmatrix} = 0 \quad \ldots\ldots (83)$$

The roots of Eq. 83 are

$$\omega^2_1 = 0.31\ D \quad \text{(84a)}$$

$$\omega^2_2 = 2.47\ D \quad \text{(84b)}$$

and

$$\omega^2_3 = 4.37\ D \quad \text{(84c)}$$

It is apparent that the proposed method yields a good approximation of the fundamental frequency (and, to a lesser extent, of the fundamental mode) of vibration. Conversely, the accuracy of the method at the high frequencies is less than the accuracy obtained by the author. This lessening of the accuracy appears reasonable if it is considered that the higher modes involve nodes which reduce the magnitude of the joint rotations.

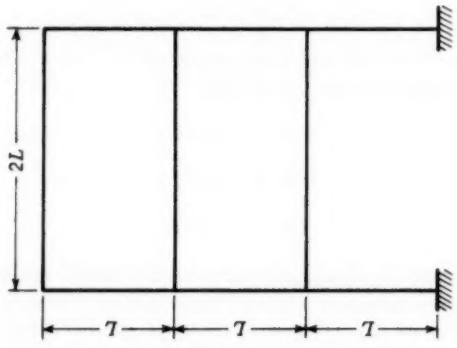

FIG. 6

EDWARD COHEN[23] AND EDWARD LAING,[24] JUNIOR MEMBERS, ASCE, AND

[23] Asst. Engr., Ammann & Whitney, New York, N. Y.

[24] Structural Designer, Ammann & Whitney, New York, N. Y.

LEON S. LEVY.[25]—A valuable contribution to the solution of dynamic problems

[25] Professional Assistant (Mathematics), Ammann & Whitney, New York, N. Y.

involving shear buildings has been made by the author. The writers have utilized the material presented by Mr. Salvadori for the dynamic analysis of structures subjected to impulsive loads. In the course of this work it was noted that the rotational mode, as defined by Eq. 36, is valid only for buildings with large height-to-width ratios where the distribution of mass across the width can be neglected. However, under these conditions, the number of actual modes and frequencies in a building subject to shear and rocking will not exceed the number of stories. As described by Mr. Bleich,[26] the

[26] "Frequency Analysis of Beam and Girder Floors," by Hans H. Bleich, *Transactions*, ASCE, Vol. 115, 1950, p. 1035.

> "*** total number [of frequencies] can include a trivial root $\omega = \infty$, however [and a mode equal to zero], which happens in the special case in which the complete system contains only masses which already form part of system B."

For structures in which the distribution of mass across the width cannot be neglected, the actual rotational modes can be derived as follows (in the same manner as the shear modes):

The derivation of the modes from Eq. 33,

$$I\ddot{\theta} + B\theta = 0 \quad \ldots\ldots (85)$$

will be valid only when the proper dimensionality has been introduced.[27]

[27] "Mathematical Methods in Engineering," by Th. von Kármán and M. A. Biot, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, p. 177.

Thus, setting x_i equal to $L_i\theta$ and dividing Eq. 85 by L_i, there results

$$\frac{L}{L^2_i}\ddot{x}_i + \frac{B}{L^2_i}x_i = 0 \cdots (i = 1,2,\cdots, N) \quad \ldots\ldots (86)$$

Eq. 86 on application of the normalizing condition for the rotational mode, leads to

$$\sum_{i=1}^{N} c^2\, x^2(i)\, \frac{I}{L^2_i} = \sum_{i=1}^{N} \phi_A{}^2(i)\, \frac{I}{L^2_i} = 1 \quad \ldots\ldots (87a)$$

from which

$$\phi_A(i) = \frac{x(i)}{\sqrt{\sum_{i=1}^{N} x^2(i)\, \frac{I}{L^2_i}}} \quad \ldots\ldots (87b)$$

Remembering that the physical condition necessitates that $x(i) = c\, L_i$,

$$\phi_A(i) = \frac{L_i}{\sqrt{I}} = \frac{L_i}{\sqrt{\sum_{i=1}^{N} M_i \left(L^2_i + \frac{b^2}{12} \right)}} \quad \ldots\ldots (88)$$

Thus, the rotational modes include a factor of the width of the building, and, as anticipated, the use of this factor reduces the rotational mode and the rota-

tion. In the illustrative example, where the building has a height-to-width ratio of 1, an error of approximately 15% is introduced by the use of Eq. 36.

It should be noted that modification of the normalizing condition (following Eq. 41*b*) and of the orthogonality condition also becomes necessary.

The writers question the validity of Mr. Salvadori's statement that

> "Inasmuch as the largest stresses due to earthquakes occur, in general, a short time after the beginning of the earthquake, and since the logarithmic decrement r, is nearly 1, the influence of damping on elastic earthquake stresses is often negligible."

For any given earthquake the maximum structural distortions occur as the result of either the severe discontinuous shocks relatively close to the epicenter or the sustained oscillations of lower acceleration and longer period which occur farther away from the epicenter.

In the case of the sustained oscillations, it is apparent that damping, even of small magnitude, must play an important part in reducing the effects of resonance for structures whose natural period approximates the period of the ground motions. However, even in the former case, damping cannot be neglected. It was found[28] that, in the case of the earthquake of May 18, 1940, at

[28] Discussion by R. R. Martel, G. W. Housner, and J. L. Alford, of "Lateral Forces of Earthquake and Wind," *Transactions*, ASCE, Vol. 117, 1952, p. 769, Fig. 17.

El Centro, Calif., the response of an idealized structure could be reduced by more than 50% by the addition of 2% of critical damping. Damping in ordinary buildings has been found to range from 7.5% to 14% of critical damping for concrete structures and 1.5% to 5% for steel structures.

MARIO G. SALVADORI,[29] A.M. ASCE.—Mr. Blume justifiably warns the

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engineer about the complexities of earthquake-resistant design, and the writer wholeheartedly agrees with him. Mr. Blume has performed a valuable service to the profession by listing (in factors 1 to 6 and (*a*) to (*k*)) some of the essential points governing the dynamics of earthquake motion in buildings. However, one must realize that the research engineer must confine his studies to the influence of some of these factors lest he be hampered by the complexity of the phenomenon. It is not a matter of controversy that the improvements in procedures stipulated in some building codes depend essentially on the results achieved by mathematical and experimental methods. The faster these results are incorporated into the codes, provided this is done intelligently, the better practical earthquake design will become.

The writer's procedures combine the classical approach with the numerical approach to obtain an increase in efficiency. The methods of Lord Rayleigh and H. Holzer are well known, but it is often a matter of familiarity that decides the designer's preference. It was gratifying to learn that Mr. Blume was able to reproduce, by other methods, the writer's numerical results.

The influence of bending on the frequencies and modes of a building, which acts primarily as a shear building, can obviously be substantial, as shown by Mr. Blume. This influence can be accounted for in the procedures suggested by the writer.

The writer is aware of the existence of other numerical and mathematical procedures which are well adapted to the solution of the earthquake problem and did not mean to imply that empirical methods should be discarded entirely.

His intention was to compare the efficiency of three methods of solving a simplified earthquake problem, in the hope of adding some knowledge to a complex field of investigation.

Mr. Masur's suggestion to account approximately for the influence of joint rotation seems to be applicable only in connection with the first mode. Since this mode can easily be obtained by Rayleigh's method, the writer does not feel that, in general, its use would be justified.

Messrs. Cohen, Laing, and Levy take issue with the writer's statement concerning the influence of damping on earthquake motion. The influence of damping on the over-all response of a building depends essentially on (1) whether there is resonance between the Fourier components of the earthquake and the natural modes of the building, and (2) the amount of energy dissipated through friction. Although it is quite possible to encounter cases in which these two factors are of fundamental importance, it is not possible to state beforehand that this should always be the case. It would be of interest to study the influence of friction on earthquake motion from a mathematical viewpoint.

Even though the influence of the width of the building is not among the most important factors in dynamic design, it can be easily taken into account by substituting for Eqs. 6*a*, 7, 36, 37, and 38, respectively, the following, in which θ_k is the rotational component of the kth mode:

$$\sum_{i=1}^{N} \psi^2_k\,(i)\,M_i + \theta^2_k \sum_{i=1}^{N} M_i \frac{b^2}{12} = 1 \qquad (89)$$

$$\sum_{i=1}^{N} \psi_k\,(i)\,\psi_j\,(i)\,M_i + \theta_k\,\theta_j \sum_{i=1}^{N} M_i \frac{b^2}{12} = 0 \qquad (90)$$

$$\phi_A\,(i) = \frac{L_i}{\sqrt{I}} \qquad (91a)$$

$$\theta_A = \frac{1}{\sqrt{I}} \qquad (91b)$$

$$\psi_k\,(i) = c_k\,\phi_A\,(i) + \sum_{j=1}^{N} c_{kj}\,\phi_{kj}\,(i) \qquad (92a)$$

$$\theta_k = c_k\,\theta_A \qquad (92b)$$

and

$$\frac{\omega^2_A}{\Omega^2_k}\,c_k = \sum_{i=1}^{N} [\phi_A\,(i)\,\psi_k\,(i)\,M_i] + \theta_k\,\theta_A \sum_{i=1}^{N} M_i \frac{b^2}{12} \qquad (93)$$

Corrections for Transactions.—In Eq. 37, k vaires from 1 to N not from 1 to $N + 1$. In Eqs. 46 and 48, the summation is from $k = 1$ to N, rather than from $k = 1$ to $N + 1$.

DISCUSSION OF STIFFNESS CHARTS FOR GUSSETED MEMBERS UNDER AXIAL LOAD
PROCEEDINGS-SEPARATE 179

LEONARD K. STEVENS.[23]—The stiffness charts presented by the author

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are very useful in the elastic analysis of the behavior of rigid-jointed trusses.

Mr. Lundquist's method of stability analysis assumes that the members of a rigid-jointed truss remain straight and are centrally loaded up to the point of failure, at which point they suddenly buckle. In actuality, the end moments introduced by the joint translations resulting from the axial deformations of the members produce curvatures, and the strut is not straight at failure. However, the ultimate load capacity of an eccentrically loaded, elastic strut is not significantly less than when loaded centrally, although large lateral deflections may occur before the maximum load is reached. For this reason, it is considered that Mr. Lundquist's method of analysis gives reasonably accurate results. Experimental work has shown good agreement in the elastic range.

It is possible that the secondary stresses produced by the end moments may lead to combined stresses approaching the yield point of the material, even though the average axial stress is still within the elastic range. If the yield point is reached in a slender strut, the load capacity decreases rapidly. The determination of the secondary stresses in a stability analysis of a truss with struts having medium slenderness ratios is therefore very important—especially when the analysis indicates that the average axial stress for limiting stability is 70% or more of the yield-point stress.

Most structures have members in the medium-to-low slenderness-ratio range. The yield point of the material is the major factor in determining the ultimate loads. As Mr. Goldberg states, plastic flow can sometimes be relied on to redistribute the critical stresses. The action when yielding has commenced in a rigid-jointed truss has not been fully investigated. For a strut having a low slenderness ratio, the appearance of initial yielding does not necessarily indicate imminent collapse; in fact, the ultimate useful load capacity can be considerably higher. An exact secondary-stress analysis can determine the load at which initial yielding begins. This load will be the lower limit of the ultimate load capacity. The upper limit is fixed by the load producing an average stress equal to the yield stress. These limits enable the collapse load to be closely bracketed. It is in this respect that the charts presented offer a valuable aid to the analysis.

It is interesting to note the effect of rigid-gusset plates and axial load on the moments found in a secondary-stress analysis. Rigid-jointed trusses with gusset plates having a ratio of a/L equal to 0.125 and strut slenderness ratios of from 55 to 110 have been analyzed by the writer. Near the load at which yielding began, the joint moments were as much as 53% greater than the moments computed by the classical methods which ignored gusset lengths and axial load.

JOHN E. GOLDBERG [24] M. ASCE.—Mr. Stevens' manifest interest in the

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stiffness charts which have been presented by the writer, and his informative and constructive comments are appreciated. His statements in support of the contention regarding the effect of gusset lengths and axial loads on the magnitude of the secondary stresses are gratifying. It is interesting and significant that Mr. Stevens, by using the stiffness charts, has found that the effect of gusset length and axial load can increase the secondary bending moments by as much as 53%, even in the case of trusses composed of members in the intermediate-slenderness range.

Mr. Stevens is entirely correct in his statement that

> "For a strut (in a rigid-jointed truss) having a low slenderness ratio, the appearance of initial yielding does not necessarily indicate imminent collapse, * * *. An exact secondary-stress analysis can determine the load at which initial yielding begins."

A rigidly connected truss is a highly redundant structure, and it appears to the writer that one can apply the limit-design concepts and can also invoke the general rule that a strut must develop three complete plastic zones before it will collapse.

Corrections for Transactions.—In the first equation of Appendix I, "P_y" should be "$P\ y$" and in Eq. 7, "P" (the term preceding the equal sign) should be "p."

DISCUSSION OF THE APPLICATION OF HEAVISIDE'S STEP-FUNCTION TO BEAM PROBLEMS.
PROCEEDINGS-SEPARATE 202

Telemaco van Langendonck.[1]—It would be more general, to define the step-function as:

$$H_a = \frac{1}{2} + \int_0^{\infty} \frac{\text{sen } t(x-a)}{\pi\, t}\, dx = \begin{cases} 0 \text{ for } x < a \\ 1 \text{ for } x > a \end{cases}$$

being of no importance the value for $x = a$, as far as it stays finite (with the given formula its value would be 0,5).

The integral (4) is not general because it is exact only when $F(0) = 0$. The general expression is (c is any fixed value of x):

$$\int_c^x H_a\, f(x)\, dx = H_a\, [F(x) - F(a)] + C.$$

The derivative dH_a/dx is not equal to zero, as appears in formula (5), unless the considered interval does not include the value $x = a$. In the same way $\frac{d}{dx} H_a\, F(x)$ only will be equal to $H_a f(x)$ if $F(a) = 0$.

With these considerations, the application of the step-function can be extended to the cases in which there are differential equations to be integrated: beams with longitudinal loads, beams on elastic foundations, etc.

Take, for instance, the case of the beam with two supports of Fig. A, with longitudinal load F and transversal load P. The equation of the elastic line is:

$$\frac{d^2y}{dx^2} + \frac{F}{EI}\, y = -\frac{M_o}{EI} \qquad (I)$$

where EI is the flexural rigidity of the cross section and M_o is the bending moment that would exist if there was not the force F:

$$M_o = \frac{Pb}{\ell}\, x - H_a\, P\, (x-a). \qquad (II)$$

Putting $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$ e $\frac{F}{EI} = k^2$, we have:

$$y'' + k^2 y = f(x).$$

In general, when f(x) is a polynomial of grade m:

$$f(x) = a_m\, x^m + a_{m-1}\, x^{m-1} + \ldots + a_2 x^2 + a_1 x + a_o$$

the solution of (III) is $y = 0_1 \text{ sen } kx + C_2 \cos kx + Y$, with:

$$Y = b_m x^m + b_{m-1}\, x^{m-1} + \ldots + b_2 x^2 + b_1 x + b_o$$

from which

$$Y'' = m(m-1)\, b_m x^{m-2} + \ldots + 6b_3 x + 2b_2.$$

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Substituting in (III) and equalizing the coefficients of the terms with equal exponents, are obtained the coefficients which permit to find Y and, therefore, y:

$$b_m = \frac{a_m}{k^2},\ b_{m-1} = \frac{a_{m-1}}{k^2},\qquad b_{m-2} = \frac{a_{m-2} - m(m-1)b_m}{k^2},\ \ldots \qquad \text{(IV)}$$

If f (x) contains step-functions

$$f(x) = f_o(x) + H_a f_a(x-a) + H_b f_b(x-b) + \ldots$$

the integration is made by considering separately each one of the terms $f_o, H_a f_a, H_a f_b, \ldots$ and by adding the found values of the Y, in order to obtain the final Y.

To integrate

$$y'' + k^2 y = H_a[a_m(x-a)^m + a_{m-1}(x-a)^{m-1} + \ldots + a_1(x-a) + a_o] \qquad \text{(V)}$$

it cannot be used, without modification, the previous process which can be applied when there is no H_a in the second member. This happens because the derivative of $H_a F(x)$ is only equal to zero if $F(a) = 0$, and this will not happen with

$$Y = b_m(x-a)^m + b_{m-1}(x-m)^{m-1} + \ldots b_1(x-a) + b_o$$

if b_o is not zero and, with Y' if b_1 is not zero.

To overcome the difficulty it is possible, therefore, substitute b_o for $b_o[1-\cos k(x-a)]$ and $b_1(x-a)$ for $b_1[(x-a) - \frac{1}{k}\operatorname{sen} k(x-a)]$. So we have

$$Y = b_m(x-a)^m + \ldots + b_2(x-2)^2 + b_1\left[(x-a) - \frac{1}{k}\operatorname{sen} k(x-a)\right] + b_o\left[1 - \cos k(x-a)\right], \qquad \text{(VI)}$$

$$Y' = m\,b_m(x-a)^{m-1} + \ldots + 2b_2(x-a) + b_1\left[1 - \cos k(x-a)\right]$$

expressions that are equal to zero when $x = a$, thus having

$$\frac{d}{dx}(H_a Y) = H_a \frac{d}{dx} Y = H_a Y' \quad \text{and} \quad \frac{d^2}{dx^2}(H_a Y) = H_a \frac{d}{dx} Y' = H_a Y''$$

and substituting in (V):

$$H_a Y'' + H_a k^2 Y = H_a[a_m(x-a)^m + \ldots + a_1(x-a) + a]$$

or

$$k^2 b_m(x-a)^m + k^2 b_{m-1}(x-a)^{m-1} + [m(m-1)b_m + k^2 b_{m-2}](x-m)^{m-2} + \ldots +$$

$$+ (6b_3 + k^2 b_1)(x-a) + (2b_2 + k^2 b_o) = a_m(x-a)^m + \ldots + a_1(x-a) + a_o$$

from which are obtained the same values of $b_m, b_{m-1}, \ldots$ found in (IV). The solution wanted for (V) is:

$$y = C_1 \operatorname{sen} kx + C_2 \cos kx + H_a Y$$

with Y given by (VI).

In the examples of the equations (I) and (II), we have:

$$y'' + k^2 y = -\frac{Pb}{EI}x + H_a \frac{P}{EI}(x-a).$$

The value of Y corresponding to the term $-\frac{Pb}{EI}x$ ($a_1 = -\frac{Pb}{EI}$, $a_o = 0$, from which $b_1 = -\frac{Pb}{EIk^2}$, $b_o = 0$):

$$Y_o = -\frac{Pb}{k^2EI}x = -\frac{Pb}{F}x$$

and the corresponding to the term $H_a\frac{P}{EI}(x-a)$ is ($a_1 = \frac{P}{EI}$, $a_o = 0$, from which $b_1 = \frac{P}{EIk^2} = \frac{P}{F}$, $b_o = 0$):

$$Y_a = \frac{P}{F}\left[(x-a) - \frac{1}{k}\operatorname{sen} k(x-a)\right]$$

from which the solution wanted

$$y = C_1 \operatorname{sen} kx + C_2 \cos kx - \frac{Pb}{Fl}x + H_a\frac{P}{F}\left[(x-a) - \frac{1}{k}\operatorname{sen} k(x-a)\right]$$

where the constants C_1 and C_2 are determined with the condition of being $y = 0$ when $x = 0$ and $x = l$, that is

$$C_2 = 0, \qquad C_1 = \frac{P}{F}\frac{\operatorname{sen} kb}{k \operatorname{sen} kl}$$

or

$$y = \frac{P}{F}\left[\frac{\operatorname{sen} kb}{k \operatorname{sen} kl}\operatorname{sen} kx - \frac{bx}{l} + H_a(x-a) - \frac{H_a}{k}\operatorname{sen} k(x-a)\right]$$

which is the equation of the elastic line of the beam of fig. I. The bending moments are

$$M = M_o + Fy = \frac{P}{k}\left[\frac{\operatorname{sen} kb}{\operatorname{sen} kl}\operatorname{sen} kx - H_a \operatorname{sen} k(x-a)\right].$$

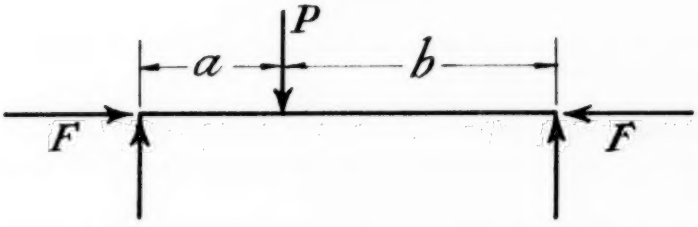

Fig.A

CORRECTION TO APPLICATIONS OF THE RELAXATION TECHNIQUE IN FLUID MECHANICS. PROCEEDINGS-SEPARATE 223

Pages 223-20 and 223-21 should have their numbers interchanged and their order of appearance reversed.

CORRECTIONS TO MECHANICS OF MANIFOLD FLOW PROCEEDINGS-SEPARATE 258

The following is to be substituted for page 258-21:

The various results are presented as functions of the ratio of lateral discharge to total and of the ratio of the sizes of the lateral and the conduit. Although the results are also dependent on the shape of the conduit and the geometry of the lateral, those for circular conduits are thought to be representative of other manifold forms. During a period of nearly half a century various experimenters have published data for a variety of other forms, so that a considerable amount of information is at hand. In this paper the writer has presented new data and systematized the presentation and interpretation of the results. It is to be emphasized that tests on a single junction are sufficient in defining the characteristics of a complete manifold system. For many installations, the results presented herein should prove to be directly applicable.

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